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Links between Prime Ideals in Differential Operator Rings

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In recent years much work has been done on a localization theory for noetherian rings. Links between prime ideals play a central role in this development. When a ring R is not commutative, it is rarely possible to localize R at a prime ideal P. Having a link between P and another prime ideal is an obstruction to localization. In this paper we will investigate the links between prime ideals in differential operator rings, showing that it is possible to localize these rings in a certain sense.

Recall that a multiplicatively closed subset S of a ring R is a *right Ore set* if for all r in R and all s in S we have $rS \cap sR \neq \emptyset$. The set S is an Ore set if it is both a right and a left Ore set. It is always possible to construct a ring RS^{-1} where the elements of S become units. However, if we insist on writing the elements of RS^{-1} as right fractions, rs^{-1} with r in R and s in S, then the set S must be a right Ore set [9, Sect. 12.1].

Let P be a prime ideal of R, and let $\mathscr{C}(P) = \{r \in R \mid r + P \text{ is regular in } R/P\}$. We say that P is a *localizable prime ideal* if the set $\mathscr{C}(P)$ is an Ore set. We will shortly give an example of a prime ideal which is not localizable, but first we introduce the differential operator ring.

Let R be a ring, and let δ be an additive map from R to itself. We call δ a *derivation* if it satisfies the product rule; $\delta(ab) = \delta(a)\, b + a\delta(b)$. The *differential operator ring* over R, denoted $R[\theta; \delta]$, is a free left R-module with basis $1, \theta, \theta^2, \ldots$. Thus every element of $R[\theta; \delta]$ is a polynomial in θ with coefficients from R. The addition in $R[\theta; \delta]$ is defined as usual for polynomials, but multiplication is extended from R by the rule $\theta r = r\theta + \delta(r)$. The differential operator ring is an associative ring and is determined up to isomorphism by an obvious universal property [9, Sect. 12.2].

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EXAMPLE. Let $R = \mathbb{C}[x]$ be the polynomials with complex coefficients, and let $\delta = x(d/dx)$. By this we mean that $\delta(f(x)) = xf'(x)$ for all $f(x)$ in R. Let $T = R[\theta; \delta]$. Note that T is the universal enveloping algebra of the solvable 2-dimensional Lie algebra. Let $P = xT + (\theta - 1)T$ and $Q = xT + \theta T$. It is clear that T/P and T/Q are both isomorphic to the complex numbers. Hence P and Q are prime ideals of T and $\mathscr{C}(P) = T - P$ and $\mathscr{C}(Q) = T - Q$. We will show that $\mathscr{C}(Q)$ is not a right Ore set, by showing that if $S \subseteq \mathscr{C}(Q)$ is a right Ore set, then $S \subseteq \mathscr{C}(P)$.

We have that $\theta x = x\theta + x$, so that $x\theta = (\theta - 1)x$ and an easy induction shows that $x\theta^i = (\theta - 1)^i x$ for $i = 1, 2, \ldots$. It now follows that if $s(\theta) \in T$, then $xs(\theta) = s(\theta - 1)x$. Note that $xT = Tx$, so xT is an ideal of T. Let $\bar{\ }$ denote the canonical map from T to T/xT. Since $\delta(R) \subseteq xR$ we get that $\bar{T} = T/xT \cong (R/xR)[\theta] \cong \mathbb{C}[\theta]$. It is now clear that $\bar{P} = (\theta - 1)\mathbb{C}[\theta]$ and $\bar{Q} = \theta\mathbb{C}[\theta]$. Take $s(\theta) \in S \subseteq \mathscr{C}(Q)$. Then $\bar{s}(\theta) \notin \bar{Q}$, whence $\bar{s}(0) \neq 0$. Since S is a right Ore set, $xs_1(\theta) = s(\theta)\, t_1(\theta)$ for some $s_1(\theta) \in S$ and $t_1(\theta) \in T$. Since $s(\theta) \notin xT$, but $s(\theta)\, t_1(\theta) \in xT$ we must have that $t_1(\theta) \in xT = Tx$. Hence $t_1(\theta) = t_2(\theta)\, x$. We now have that $s_1(\theta - 1)\, x = s(\theta)\, t_2(\theta)\, x$, and since T is a domain $s_1(\theta - 1) = s(\theta)\, t_2(\theta)$. From the observation above, $0 \neq \bar{s}_1(0) = \bar{s}(1)\, \bar{t}_2(1)$, and thus $\bar{s}(1) \neq 0$. It follows that $s(\theta) \in T - P = \mathscr{C}(P)$ and thus $S \subseteq \mathscr{C}(P)$.

It turns out that, in the above example, the prime ideals P and Q are linked (see Example 5.7, or [5, Example 1.3]). We now define what it means for two prime ideals to be linked, and state a lemma which shows how links are an obstruction to localization. We refer the reader to [12] for a proof of the lemma, and to Jategaonkar's memoir [13] for further information.

DEFINITION. Let R be a right noetherian ring and let P and Q be prime ideals of R. We say that P is *linked* to Q, denoted $P \rightsquigarrow Q$, if there is an ideal I of R such that $PQ \subseteq I \subset P \cap Q$ and $P \cap Q/I$ is torsion-free as a left R/P-module and as a right R/Q-module. A link from P to itself will be called a *trivial link*. The *link graph* of R is the directed graph whose vertices are the elements of Spec R, and there is a directed edge from P to Q if and only if P is linked to Q. The *right clique* of a prime Q, denoted $\underrightarrow{\Omega}_Q$, is the smallest set of prime ideals which contains Q, and is such that if P_2 is in $\underrightarrow{\Omega}_Q$ and $P_1 \rightsquigarrow P_2$, then P_1 is in $\underrightarrow{\Omega}_Q$. The *left clique* of Q, denoted ${}_Q\underrightarrow{\Omega}$, is the smallest set of prime ideals which contains Q, and is such that if P_1 is in ${}_Q\underrightarrow{\Omega}$ and $P_1 \rightsquigarrow P_2$, then P_2 is in ${}_Q\underrightarrow{\Omega}$. The *clique* of Q, denoted Ω_Q or just Ω, is the connected component of the link graph that contains Q.

LEMMA. *Let R be a noetherian ring with prime ideals P and Q, and let $S \subseteq \mathscr{C}(Q)$ be a right Ore set. If P is linked to Q, then $S \subseteq \mathscr{C}(P)$.*

DEFINITION. For $X \subseteq \operatorname{Spec} R$, let $\mathscr{C}(X) = \bigcap_{P \in X} \mathscr{C}(P)$.

In view of the lemma, it is clear that if Q is a prime ideal of R and S is a right Ore set contained in $\mathscr{C}(Q)$, then S is contained in $\mathscr{C}(\underset{\rightarrow}{\Omega}_Q)$. Thus if $\mathscr{C}(\underset{\rightarrow}{\Omega}_Q)$ is a right Ore set, then it is the largest right Ore set contained in $\mathscr{C}(Q)$.

The purpose of this paper is to determine the link graph of a differential operator ring $R[\theta; \delta]$, when R is a commutative noetherian ring containing the rational numbers. We will also show that the set $\mathscr{C}(\underset{\rightarrow}{\Omega}_Q)$ is always a right Ore set. Moreover, we will show how the link graph can be used to determine the structure of certain injective hulls. Some of these same questions have been studied for universal enveloping algebras (Brown in [6] and [7]), and for skew polynomial rings, $R[x; \sigma]$ where σ is an automorphism (Poole in [17]).

In this paper all rings will have an identity, and all modules will be unitary. We will also assume that all multiplicatively closed sets contain the identity element of the ring. Modules will be considered to be right modules unless we specify otherwise. For a module M, we let $E(M)$ (or $E_R(M)$ if it is necessary to specify the ring R) denote the injective hull of M. When we say "noetherian," we mean right and left noetherian. Similarly for other properties like "ideal" and "Ore." When we write $A \subset B$, we always mean that A is a proper subset of B. Let B be a subset of an R-module A. We let $r(B) = \{r \in R \mid Br = 0\}$ denote the right annihilator of B. If A is a left R-module, $\ell(B)$ will denote the left annihilator of B.

The elements of $T = R[\theta; \delta]$ will usually be written as $f(\theta)$, including the variable θ, while elements of R will be denoted as r. However, in some proofs we will drop the variable θ in order to avoid very long expressions.

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1. PRELIMINARIES

Let R be a ring, and let δ be a derivation on R. If there is an element d of R such that $\delta(r) = dr - rd$ for all r in R we say that δ is an *inner derivation.* An ideal I of R satisfying $\delta(I) \subseteq I$ is called a *δ-ideal,* and we say that R is *δ-simple* if 0 and R are the only δ-ideals. Let I be a δ-ideal. We call I a *δ-prime* ideal if $I \neq R$, and for all δ-ideals A and B of R with $AB \subseteq I$, we have $A \subseteq I$ or $B \subseteq I$. The following lemma lists some elementary facts about the differential operator ring. For a proof we refer the reader to [14].

LEMMA 1.1. *Let R be a ring with a derivation δ, and let $T = R[\theta; \delta]$.*

(i) *If I is a δ-ideal of R, then $IT = TI$, so IT is an ideal of T.*

(ii) *If J is an ideal of T, then $J \cap R$ is a δ-ideal of R.*

(iii) *If J is a prime ideal of T, then $J \cap R$ is a δ-prime ideal of R.*

(iv) *If I is a δ-prime ideal of R, then IT is a prime ideal of T.*

(v) *If δ is an inner derivation, $\delta(r) = dr - rd$ for all r in R, then $R[\theta; \delta] = R[x]$, where $x = \theta - d$.*

(vi) *Let I be a δ-ideal of R. Let $\bar{T} = T/IT$ and $\bar{R} = R/I$ and $\bar{\delta}(r + I) = \delta(r) + I$. Then $\bar{\delta}$ is a derivation on $\bar{R}$, and $\bar{T} \cong \bar{R}[\theta; \bar{\delta}]$.*

It follows from (i) and (v) that if $R[\theta; \delta]$ is to be a simple ring, then R has to be a δ-simple ring, and δ must not be an inner derivation. The following proposition states that these conditions are sufficient when R is a $\mathbb{Q}$-algebra. A proof can be found in [10, Theorem 3.2].

PROPOSITION 1.2. *Let R be a $\mathbb{Q}$-algebra. $R[\theta; \delta]$ is a simple ring if and only if R is δ-simple and δ is not inner.*

PROPOSITION 1.3. *Let R be a right noetherian ring containing the rational numbers. If P is a prime ideal of $R[\theta; \delta]$, then $R \cap P$ is a prime ideal of R.*

Remark. The proposition is proved in [11, Corolarry 1.4] or [3, Satz 4.2]. We will use this result repeatedly, usually without referring to it.

PROPOSITION 1.4. *Let R be a ring with a derivation δ and let $T = R[\theta; \delta]$. If S is a right Ore set in R, then S is a right Ore set in T. Moreover, δ extends uniquely to a derivation δ_S on RS^{-1}, given by $\delta_S(rs^{-1}) = \delta(r)\, s^{-1} - rs^{-1}\delta(s)\, s^{-1}$, and $TS^{-1} \cong RS^{-1}[\theta; \delta_S]$.*

It is not difficult to prove the proposition by directly verifying the Ore condition. An indirect proof for the case when S consists of regular elements is given in [3, Satz 4.4]. We will often use the proposition when R is a commutative ring, M is a prime ideal of R and $S = R - M$. In this case we will let R_M, T_M, and δ_M denote RS^{-1}, TS^{-1}, and δ_S, respectively.

We will now turn our attention to classifying the prime ideals of $R[\theta; \delta]$, when R is a commutative ring containing the rational numbers.

LEMMA 1.5. *Let R be a commutative $\mathbb{Q}$-algebra, and let M be a maximal δ-ideal of R. If MT is not a maximal ideal of $R[\theta; \delta]$, then $\delta(R) \subseteq M$.*

Proof. Using Lemma 1.1 we have that $\bar{T} \cong \bar{R}[\theta; \bar{\delta}]$ where $\bar{T} = T/MT$ and $\bar{R} = R/M$ and $\bar{\delta}(r + M) = \delta(r) + M$. By assumption $\bar{T}$ is not a simple ring, but $\bar{R}$ is δ-simple. Therefore δ must be inner by Proposition 1.2. Since R is commutative we have that $\delta(R) \subseteq M$. ∎

LEMMA 1.6. *Let R be a commutative noetherian ring containing the rational numbers. Let P and Q be prime ideals of $T = R[\theta; \delta]$ such that $P \cap R = Q \cap R = M$. If $Q \subset P$, then $Q = MT$ and $\delta(R) \subseteq M$. Moreover, if M is a maximal ideal of R, then $P = MT + p(\theta)\, T$ for some $p(\theta) \in T$, and $\bar{p}(\theta) \in (R/M)[\theta]$ is an irreducible polynomial.*

Proof. We localize R at the prime ideal M, producing $T_M \cong R_M[\theta; \delta_M]$, and $Q_M \subset P_M$. Thus $M_M T_M$ is not a maximal ideal of T_M. Since M_M is a maximal ideal of R_M, we can apply Lemma 1.5 to get that $\delta_M(R_M) \subseteq M_M$. If $r \in R$, then $\delta(r) \in M_M$. Hence $\delta(r)\, s \in M$ for some $s \in R - M$, and thus $\delta(R) \subseteq M$.

Let $\bar{T} = T/MT$ and $\bar{R} = R/M$. Then $\bar{T} = \bar{R}[\theta]$ and $\bar{P} \cap \bar{R} = \bar{Q} \cap \bar{R} = 0$. Since $\bar{Q}$ is properly contained in $\bar{P}$, it follows that $\bar{Q} = 0$ [15, Theorem 36]. If M is a maximal ideal, then $\bar{R}$ is a field and $\bar{P}$ is a non-zero prime ideal in the principal ideal domain $\bar{R}[\theta]$. ∎

2. AR-IDEALS AND AR-SEPARATED RINGS

Much of the material in this section is known in greater generality than what we present here, see [1] and [2]. For the sake of completeness we include the proofs of the specific results needed.

An ideal I in a ring R is said to be a *right AR-ideal* if for every right ideal K, there exists an integer n such that $K \cap I^n \subseteq KI$. An ideal is an *AR-ideal* if it is both a left and a right AR-ideal. A ring R is a *[right] AR-ring* if every ideal of R is a [right] AR-ideal. A ring R is *[right] AR-separated* if for every pair of prime ideals P and Q such that $P \subset Q$, there exists an ideal I such that; $P \subset I \subseteq Q$ and I/P is a [right] AR-ideal in R/P. The following Proposition and its proof is extracted from [4, Theorem 1, p. 196].

PROPOSITION 2.1. *Let I be an ideal of a ring R, and let*

$$R' = R \oplus Ix \oplus I^2x^2 \oplus \cdots \oplus I^nx^n \oplus \cdots.$$

If R' is a right noetherian ring, then I is a right AR-ideal.

Proof. If K is a right ideal of R, then

$$K' = K \oplus (K \cap I)\, x \oplus (K \cap I^2)\, x^2 \oplus \cdots \oplus (K \cap I^n)\, x^n \oplus \cdots$$

is a right ideal of R'. Since R' is right noetherian, K' is finitely generated, and K' can be generated by elements of the form; $u_i x^{n(i)}$, for $i = 1, 2, \ldots, m$, with $u_i \in K \cap I^{n(i)}$. Let N be the largest of the integers $n(1), n(2),..., n(m)$. If $n \geqslant N$ and $u \in K \cap I^n$, then $ux^n = \sum_{i=1}^m u_i x^{n(i)} v_i$ for some $v_1,..., v_m \in R'$. We may assume that $v_i = a_i x^{n-n(i)}$ with $a_i \in I^{n-n(i)}$. Hence $ux^n = [\sum_{i=1}^m u_i a_i]\, x^n$, and thus

$$u = \sum_{i=1}^{m} u_i a_i \in (K \cap I^{n(i)})\, I^{n-n(i)} = (K \cap I^{n(i)})\, I^{N-n(i)} I^{n-N} \subseteq (K \cap I^N)\, I^{n-N}.$$

Since the reverse inclusion is obvious, we have that $K \cap I^n = (K \cap I^N)\, I^{n-N}$. Therefore

$$K \cap I^{n+1} = (K \cap I^N)\, I^{n+1-N} = (K \cap I^n)\, I \subseteq KI. \quad \blacksquare$$

LEMMA 2.2. *Let R be a commutative noetherian ring and I an ideal of R. Then $R' = R \oplus Ix \oplus I^2x^2 \oplus \cdots \oplus I^n x^n \oplus \cdots$ is a noetherian ring.*

Proof. Since R is noetherian, I is finitely generated. Let $\{m_1,..., m_k\}$ be a set of generators. Define a map from $R[x_1,..., x_k]$ to R' by $p(x_1,..., x_k) \mapsto p(m_1 x,..., m_k x)$. This is a ring homomorphism because R is commutative. It is clear that this map is onto. It follows that R' is noetherian. $\blacksquare$

PROPOSITION 2.3. *Let $T = R[\theta; \delta]$, where R is a commutative noetherian ring. If I is a δ-ideal of R, then IT is an AR-ideal of T.*

Proof. Let R' be as above, and let

$$T' = T \oplus ITx \oplus I^2Tx^2 \oplus \cdots \oplus I^nTx^n \oplus \cdots.$$

Extend δ to R' by letting $\delta(x) = 0$. It is easy to verify that $T' \cong R'[\theta; \delta]$. By Lemma 2.2, R' is noetherian, so T' is noetherian. It now follows from Proposition 2.1 that IT is an AR-ideal. $\blacksquare$

PROPOSITION 2.4. *Let R is a commutative noetherian ring containing the rational numbers. Then $R[\theta; \delta]$ is an AR-separated ring.*

Proof. Let $P \subset Q$ be prime ideals in $T = R[\theta; \delta]$. There are two cases. First, assume $P \cap R \subset Q \cap R$. As we have seen, $(Q \cap R)\, T$ is an AR-ideal. Thus $((Q \cap R)\, T + P)/P$ is an AR-ideal in T/P [8, Corollary 11.5]. Since $P \cap R \neq Q \cap R$, we have that $P \neq (Q \cap R)\, T + P$. The other case is when $P \cap R = Q \cap R$. By Lemma 1.6 we must have that $P = (P \cap R)\, T$ and $\delta(R) \subseteq R \cap P$. Thus $T/P \cong (R/P \cap R)[\theta]$, so T/P is a commutative ring. Therefore Q/P is an AR-ideal in T/P. $\blacksquare$

DEFINITION. Let P and Q be prime ideals in a noetherian ring R. We say that there is an *ideal link* from P to Q, if there are ideals $A \subset B$ of R

such that $PB \subseteq A$, $BQ \subseteq A$ and B/A is torsion-free both as a left R/P-module and as a right R/Q-module.

LEMMA 2.5. *Let R be an AR-separated noetherian ring and let P and Q be prime ideals of R such that $P \subseteq Q$. If there is an ideal link between P and Q, then $P = Q$.*

Proof. Assume there is an ideal link from P to Q. Let I be an ideal of R such that $P \subseteq I \subseteq Q$ and I/P is a right AR-ideal in R/P. We prove that $P = I$, thus showing that $P = Q$. There are ideals $A \subset B$ such that $r(B/A) = Q$ and $\ell(B/A) = P$. There is an integer n, such that

$$(I/P)^n(B + P/P) \subseteq (B + P/P) \cap (I/P)^n \subseteq (B + P/P)(I/P)$$
$$\subseteq (BI + P)/P \subseteq (A + P)/P.$$

It follows that $(I/P)^n(B/A) = 0$, and thus $(I/P)^n = 0$. Since P is a prime ideal $P = I$. The case when there is an ideal link from Q to P is proved similarly, using the fact that R is left AR-separated. ∎

The following theorem is an immediate consequence of [20, Lemma 5 and Theorem 6].

THEOREM 2.6. *Let R be a noetherian ring with nil radical N, and assume N is prime. Let P and Q be prime ideals of R. The following are equivalent.*

(i) *$\mathscr{C}$ is an Ore set.*

(ii) *If there is an ideal link from P to Q, then $P = N$ or $Q = N$ implies $P = Q = N$.*

Remark. If R is an AR-separated ring, then Lemma 2.5 shows that condition (ii) holds.

THEOREM 2.7. *Let R be a commutative noetherian ring, and let $T = R[\theta; \delta]$. If I is a δ-prime ideal of R, then IT is a localizable prime ideal of T.*

Proof. Since IT is an AR-ideal we can apply [19, Proposition 2.1]. Thus it is sufficient to show that IT/I^nT is localizable in T/I^nT for all n. The nil radical of T/I^nT is precisely IT/I^nT. Since $T/I^nT \cong (R/I^n)[\theta; \delta]$, the ring T/I^nT is AR-separated. We can thus use Theorem 2.6 to conclude that IT/I^nT is localizable. ∎

In view of this theorem and the lemma in the introduction, an ideal of the form IT in $T = R[\theta; \delta]$, where I is a δ-ideal of R, has no nontrivial links.

LEMMA 2.8. *Let P and Q be prime ideals in R, and suppose there is an ideal link from P to Q. Let I be a left AR-ideal with $I \subseteq P$. Then $I \subseteq Q$.*

Proof. Let $A \subseteq B$ be ideals of R such that $\ell(B/A) = P$, and $r(B/A) = Q$. There is an integer n such that $BI^n \subseteq I^n \cap B \subseteq IB \subseteq A$. Thus $I^n \subseteq Q$, and since Q is a prime ideal, $I \subseteq Q$. ▮

LEMMA 2.9. *Let S be a right Ore set in a prime right Goldie ring R. Either S consists of regular elements, or $0 \in S$.*

Proof. Assume $0 \notin S$, and let $I = \{r \in R \mid rs = 0 \text{ for some } s \in S\}$. Using the right Ore condition it is easy to check that I is an ideal of R. In a prime ring any nonzero ideal is essential as a right ideal, and in a prime right Goldie ring any essential right ideal contains a regular element. Since I contains no regular elements, we conclude that $I = 0$. ▮

It is easy to prove the following lemma using Lemma 2.9.

LEMMA 2.10. *Let R be a prime right Goldie ring, and let S be a right Ore set of R such that $0 \notin S$. Let M be a torsion-free R-module. Then MS^{-1} is a torsion-free RS^{-1}-module.*

LEMMA 2.11. *Let P and Q be prime ideals in a noetherian ring R. Let S be a right Ore set disjoint from both P and Q. Then P is linked to Q if and only if PS^{-1} is linked to QS^{-1}.*

Proof. The last lemma shows that if $P \rightsquigarrow Q$, then $PS^{-1} \rightsquigarrow QS^{-1}$. Conversely, let I be an ideal of RS^{-1} such that $PS^{-1}QS^{-1} \subseteq I \subset PS^{-1} \cap QS^{-1}$ and $PS^{-1} \cap QS^{-1}/I$ is torsion-free both as a right RS^{-1}/QS^{-1}-module and as a left RS^{-1}/PS^{-1}-module. Let $J = I \cap R$. Take $x \in PS^{-1} \cap QS^{-1}$ such that $x \notin I$. Then $x = ps^{-1} = qs^{-1}$ for some $p \in P$ and $q \in Q$ and $s \in S$. Thus $xs \in P \cap Q$. If $xs \in I$, then $s \in QS^{-1}$ which is impossible. Therefore $xs \notin J$ and hence $PQ \subseteq J \subset P \cap Q$. Take $u \in P \cap Q$ such that $u \notin J$, and take $r \in R$ such that $ur \in J$. Then $r \in QS^{-1}$, and thus $rs \in Q$ for some $s \in S$. It follows from Lemma 2.9 that $r \in Q$. To show that $P \cap Q/J$ is torsion-free as a left R/P-module is similar. ▮

The last lemma enables us to make the following reduction when dealing with the differential operator ring $T = R[\theta; \delta]$. Let P and Q be two prime ideals of T. We know that $P \cap R$ and $Q \cap R$ are prime ideals of R. Moreover, if we assume that P is linked to Q, then $P \cap R = Q \cap R = M$ by Lemma 2.8. Since $R - M$ is an Ore set in T disjoint from both P and Q, we can localize the coefficient ring R at M, and thus without loss of generality assume that M is a maximal ideal of R.

3. Links between Maximal Ideals in $R[\theta; \delta]$

Let $T = R[\theta; \delta]$, where R is a commutative noetherian ring. Let M be a maximal ideal of R such that $\delta(R) \subseteq M$. Let $P = MT + p(\theta)\, T$ be a maximal ideal of T. In this section, we will find all prime ideals Q such that P is linked to Q. In the previous sections we showed that when R contains the rational numbers, the problem of computing links in $R[\theta; \delta]$ can always be reduced to the situation described above. In this section, however, it will not be necessary in most cases to assume that R contains the rational numbers. If P is linked to Q, then by Lemma 2.8 $Q \cap R = P \cap R = M$. And since MT has no nontrivial links, Q must be of the form $Q = MT + q(\theta)\, T$ for some $q(\theta) \in T$.

We will use $\bar{\ }$ to denote the natural map from R to R/M and also to denote the natural map from T to T/MT. Therefore $\bar{T} = T/MT$ and $\bar{R} = R/M$. By Lemma 1.6, $\bar{T} \cong \bar{R}[\theta]$, and $\bar{p}(\theta)$ and $\bar{q}(\theta)$ are irreducible polynomials over the field $\bar{R}$.

Lemma 3.1. *Let $P = MT + p(\theta)\, T$ and $Q = MT + q(\theta)\, T$ be distinct maximal ideals of $R[\theta; \delta]$, where $M = P \cap R = Q \cap R$. Then*

(i) $P \cap Q = MT + p(\theta)\, q(\theta)\, T.$

(ii) $PQ = M^2T + p(\theta)\, MT + Mq(\theta)\, T + p(\theta)\, q(\theta)\, T.$

Proof. Part (i) is immediate because $\bar{P} \cap \bar{Q} = \bar{p}(\theta)\, \bar{q}(\theta)\, \bar{T}$. Let

$$I = M^2T + p(\theta)\, MT + Mq(\theta)\, T + p(\theta)\, q(\theta)\, T.$$

Note that $I \subseteq PQ$. To prove the reverse inclusion, it suffices to show that for v_1, v_2 in MT, and t_1, t_2 in T we have

$$[v_1 + pt_1][v_2 + qt_2] = v_1 v_2 + v_1 q t_2 + p t_1 v_2 + p t_1 q t_2 \in I$$

It is therefore sufficient to show that elements of the form $m\theta^n q(\theta)$ and $p(\theta)\, r\theta^n q(\theta)$, where $m \in M$ and $r \in R$ and $n \in \mathbb{N}$, belong to I. This follows from the communication rule $\theta^n s = s\theta^n + \sum_{i=1}^{n} \binom{n}{i} \delta^i(s)\, \theta^{n-i}$ and the fact that $\delta^i(s) \in M$ if $i \geqslant 1$. ∎

Remark. Let P and Q be as in Lemma 3.1. Note that if $f(\theta) \in R[\theta; \delta]$ and $Mf(\theta) \subseteq PQ$, then $(P \cap Q)\, f(\theta) \subseteq PQ$ (remember that $MT = TM$). Thus if P is linked to Q and if $Mf(\theta) \subseteq PQ$, then $f(\theta) \in \imath(P \cap Q/PQ) = Q$.

It will be necessary to introduce some more notation. We let $\hat{\ }$ denote the natural map from R to R/M^2, and also the natural map from T to T/M^2T. Thus $\hat{M} = M/M^2$, and $\widehat{MT} = MT/M^2T$, and we let $\hat{\delta}$: $\hat{M} \to \hat{M}$

denote the map $\hat{\delta}(m+M^2)=\delta(m)+M^2$. It is clear that $\hat{M}$ is a vector space over $\bar{R}$, and since $\delta(R)\subseteq M$ we get for $\bar{r}\in\bar{R}$ and $\hat{m}\in\hat{M}$ that

$$\hat{\delta}(\bar{r}\hat{m})=\delta(rm)+M^2=\delta(r)\,m+r\delta(m)+M^2=\bar{r}\hat{\delta}(\hat{m}).$$

Thus $\hat{\delta}$ is a linear transformation on $\hat{M}$. It turns out that the linear transformation $\hat{\delta}$ determinates completely, for a given prime P, all the prime ideals Q such that P is linked to Q.

Let $\{\hat{m}_1,\dots,\hat{m}_n\}$ be a basis for $\hat{M}$ over $\bar{R}$. The matrix of $\hat{\delta}$ over $\bar{R}$ with respect to this basis is $U=(u_{ij})$ if and only if $\hat{\delta}(\hat{m}_j)=\sum_{i=1}^n \hat{m}_i u_{ij}$. Thus we are letting $n\times n$ matrices act on column vectors on the left. We will let I_n denote the $n\times n$ identity matrix.

LEMMA 3.2. *Let A be an $n\times n$ matrix over a commutative ring R, and t an element of R. Then* $\det(tI_n+A)=\mathrm{tr}+\det A$ *for some r in R.*

Proof. Let $f(x)$ be the characteristic polynomial of $-A$. Then $f(x)=\det(xI_n+A)$, so $f(0)=\det A$. Thus $f(x)-\det A=xg(x)$ for some $g(x)\in R[x]$. Let $x=t$ and $r=g(t)$. Then $\det(tI_n+A)=\mathrm{tr}+\det A$. ∎

PROPOSITION 3.3. *Let U be a matrix for $\hat{\delta}$ over $\bar{R}$. Let $P=MT+p(\theta)\,T$ and $Q=MT+q(\theta)\,T$ be distinct maximal ideals of $T=R[\theta;\delta]$, and assume P is linked to Q. Then* $\det\bar{p}(\theta I_n+U)\in\bar{Q}$, *or equivalently, $\bar{q}(\theta)$ divides* $\det\bar{p}(\theta I_n+U)$.

Proof. Let $m_1,\dots,m_n$ be elements of M such that $\hat{m}_1,\dots,\hat{m}_n$ forms a basis for $\hat{M}$ over $\bar{R}$, and such that U is the matrix of $\hat{\delta}$ relative to this basis. Let $\hat{\mathbf{m}}=(\hat{m}_1,\dots,\hat{m}_n)$. From the equation $\theta m_j=m_j\theta+\delta(m_j)$ we get that, $\theta\hat{m}_j=\hat{m}_j\theta+\hat{\delta}(\hat{m}_j)=\hat{m}_j\theta+\sum_{i=1}^n \hat{m}_i u_{ij}$. It follows that $\theta\hat{\mathbf{m}}=\hat{\mathbf{m}}\theta+\hat{\mathbf{m}}U=\hat{\mathbf{m}}(\theta I_n+U)$. An easy induction shows that $\theta^k\hat{\mathbf{m}}=\hat{\mathbf{m}}(\theta I_n+U)^k$. Let $f(\theta)=\sum_{k=0}^s a_k\theta^k\in T$. Then

$$\bar{f}(\theta)\,\hat{\mathbf{m}}=\sum_{k=0}^{s}\bar{a}_k\theta^k\hat{\mathbf{m}}=\hat{\mathbf{m}}\sum_{k=0}^{s}\bar{a}_k(\theta I_n+U)^k=\hat{\mathbf{m}}\bar{f}(\theta I_n+U).$$

It is clear that $\bar{p}(\theta)\,\hat{\mathbf{m}}+\hat{\mathbf{m}}\bar{q}(\theta)$ is an n-tuple with all its coordinates in $\widehat{PQ}$. By using the above equation, we get that

$$\bar{p}(\theta)\,\hat{\mathbf{m}}+\hat{\mathbf{m}}\bar{q}(\theta)=\hat{\mathbf{m}}(\bar{p}(\theta I_n+U)+\bar{q}(\theta)\,I_n).$$

Hence $\hat{\mathbf{m}}(\bar{p}(\theta I_n+U)+\bar{q}(\theta)\,I_n)$ is an n-tuple with all its coordinates in $\widehat{PQ}$. For a matrix A, let A^* denote the classical adjoint matrix, and recall that $AA^*=(\det A)\,I_n$. Thus

$$\hat{\mathbf{m}}(\bar{p}(\theta I_n+U)+\bar{q}(\theta)\,I_n)(\bar{p}(\theta I_n+U)+\bar{q}(\theta)\,I_n)^*$$
$$=\hat{\mathbf{m}}\det(\bar{p}(\theta I_n+U)+\bar{q}(\theta)\,I_n)$$

Hence $\hat{m}_i \det(\bar{p}(\theta I_n + U) + \bar{q}(\theta) I_n)$ is an element of $\widehat{PQ}$ for $i = 1, 2, \ldots, n$. Therefore $\hat{M} \det(\bar{p}(\theta I_n + U) + \bar{q}(\theta) I_n) \subseteq \widehat{PQ}$. From the remark following Lemma 3.1, we get that $\det(\bar{p}(\theta I_n + U) + \bar{q}(\theta) I_n) \in \bar{Q}$. By Lemma 3.2,

$$\det(\bar{p}(\theta I_n + U) + \bar{q}(\theta) I_n) = \bar{q}(\theta)\, \bar{u}(\theta) + \det \bar{p}(\theta I_n + U)$$

for some $u(\theta) \in T$, whence $\det \bar{p}(\theta I_n + U) \in \bar{Q}$. ∎

Our next goal is to prove the converse of Proposition 3.3. We will show that if $\bar{q}(\theta)$ is an irreducible factor of $\det \bar{p}(\theta I_n + U)$, then $P = MT + p(\theta)\, T$ is linked to $Q = MT + q(\theta)\, T$. We will use the following lemma to show that two maximal ideals are linked.

LEMMA 3.4. *Let R be a noetherian ring, and let P and Q be maximal ideals of R such that R/P and R/Q are artinian. If there is a nonsplit exact sequence*

$$0 \longrightarrow R/Q \xrightarrow{\psi} A \xrightarrow{\pi} R/P \longrightarrow 0$$

of right R-modules, then P is linked to Q.

Proof. We will show that $P \cap Q/PQ$ is torsion-free as a left R/P-module as well as a right R/Q-module. To prove this, it is sufficient, because R/P and R/Q are simple artinian rings, to show that $r(P \cap Q/PQ) = Q$ and $\ell(P \cap Q/PQ) = P$. Since P and Q are maximal ideals, the last statement holds if we simply show that $PQ \neq P \cap Q$.

Let $a \in A$ and $p \in P$. Then $\pi(ap) = \pi(a)\, p = 0$. Thus $ap \in \operatorname{Ker} \pi = \operatorname{Im} \psi$, so $\psi(b) = ap$ for some $b \in T/Q$. Let $q \in Q$. Then $apq = \psi(b)\, q = \psi(bq) = 0$. Therefore $APQ = 0$. Suppose $A(P \cap Q) = 0$. Since $P \cap Q$ annihilates both R/P and R/Q, we may consider the exact sequence as a sequence of right $R/P \cap Q$-modules. Since $R/P \cap Q$ is a semisimple artinian ring, the sequence must split, contradicting the assumption of the lemma. Hence $A(P \cap Q) \neq 0$, and thus $PQ \neq P \cap Q$. ∎

So far we have been assuming that P and Q are distinct maximal ideals. It is easy to see that a maximal ideal $P = MT + p(\theta)\, T$ of $T = R[\theta; \delta]$ is always linked to itself. To prove this it is enough to show that $P \neq P^2$ (see the beginning of the proof of Lemma 3.4). If $P = P^2$, then

$$p \in P^2 = M^2T + MpT + pMT + p^2T = M^2T + MTp + pMT + p^2T$$

and can therefore be written as $p = p^2t + up + pv + m$, where u and v are in MT and m is in M^2T and t is in T. Thus $p(1 - pt) \in MT$, which means that $1 - pt \in MT$ which is impossible.

To prove the converse of Proposition 3.3, we will work with the $T-T$ bimodule $\widehat{MT}=MT/M^2T$. Since $\widehat{MT}$ is annihilated on both sides by M, $\widehat{MT}$ can be viewed as a $\bar{R}[\theta]$-bimodule. Even though $\bar{R}[\theta]$ is a commutative ring, the left and right actions of $\bar{R}[\theta]$ on $\widehat{MT}$ are not the same. To see this let $m \in M$. Then $\theta\hat{m}=\theta m+M^2T=m\theta+\delta(m)+M^2T$ and $\hat{m}\theta=m\theta+M^2T$. It is clear that $\theta\hat{m}=\hat{m}\theta$ only if $\delta(m)\in M^2$. When we consider $\widehat{MT}$ as a $\bar{R}[\theta]$-module, we will always mean the right action of $\bar{R}[\theta]$. The left action of $\bar{R}[\theta]$ will be used to induce right module maps of $\widehat{MT}$. It is easy to see that $\widehat{MT}$ is a free $\bar{R}[\theta]$-module with basis $\{\hat{m}_1,\ldots,\hat{m}_n\}$ (we mean the same basis as in Proposition 3.3, but now $\hat{m}=m+M^2T$).

LEMMA 3.5. *Let $p(\theta)$ be an element of T, and let Δ_p be the endomorphism of $\widehat{MT}$ that has matrix $\bar{p}(\theta I_n+U)$ relative to the basis $\{\hat{m}_1,\ldots,\hat{m}_n\}$. Then $\Delta_p(\hat{v}(\theta))=\bar{p}(\theta)\,\hat{v}(\theta)$ for all $\hat{v}(\theta)$ in $\widehat{MT}$.*

Proof. Let Δ be the endomorphism with matrix θI_n+U relative to the same basis. Let e_j be an n-dimensional column vector with 1 in the jth entry and zeros elsewhere. Then $\Delta(\hat{m}_j)=\sum_{i=1}^n \hat{m}_i\bar{r}_i(\theta)$ if and only if $(\theta I_n+U)\,e_j=\sum_{i=1}^n e_i\bar{r}_i(\theta)$. Since $(\theta I_n+U)\,e_j=e_j\theta+\sum_{i=1}^n e_iu_{ij}$, we have that

$$\Delta(\hat{m}_j)=\hat{m}_j\theta+\sum_{i=1}^{n}\hat{m}_iu_{ij}=\hat{m}_j\theta+\hat{\delta}(\hat{m}_j)=\theta\hat{m}_j.$$

Let $\hat{v}(\theta)\in\widehat{MT}$. Then $\hat{v}(\theta)=\sum_{j=1}^n \hat{m}_j\bar{v}_j(\theta)$ for some $\bar{v}_j\in\bar{R}[\theta]$, so $\Delta(\hat{v}(\theta))=\theta\hat{v}(\theta)$. Using induction on k, we get that $\Delta^k(\hat{v}(\theta))=\theta^k\hat{v}(\theta)$. Let $\bar{p}(\theta)=\sum_{k=0}^s \bar{a}_k\theta^k$. Then $\Delta_p=\sum_{k=0}^s \bar{a}_k\Delta^k$, whence $\Delta_p(\hat{v}(\theta))=\bar{p}(\theta)\,\hat{v}(\theta)$. ∎

PROPOSITION 3.6. *Let $p(\theta)$ be an element of T. Let $U\in\mathrm{Mat}_n\bar{R}$ be a matrix for $\hat{\delta}$, and let $\bar{q}(\theta)$ be an irreducible factor of* det $\bar{p}(\theta I_n+U)$. *There exist polynomials $\bar{g}_1(\theta),\ldots,\bar{g}_n(\theta)$ in $\bar{R}[\theta]$ having the following properties:*

(1) *$\bar{g}_i(\theta)$ divides $\bar{g}_{i+1}(\theta)$ and* det $\bar{p}(\theta I_n+U)=\bar{g}_1(\theta)\cdots\bar{g}_n(\theta)$. *Thus every irreducible factor of* det $\bar{p}(\theta I_n+U)$ *divides $\bar{g}_n(\theta)$.*

(2) *$\widehat{MT}\bar{g}_n(\theta)\subseteq\bar{p}(\theta)\,\widehat{MT}$, and $\bar{g}_n(\theta)$ divides any polynomial $\bar{h}(\theta)$ that has this property.*

Now assume $\bar{p}(\theta)$ is irreducible in $\bar{R}[\theta]$, and let $P=MT+p(\theta)\,T$ and $Q=MT+q(\theta)\,T$.

(3) *Let $\bar{g}_k(\theta)$ be the first one of the polynomials $\bar{g}_1(\theta),\ldots,\bar{g}_n(\theta)$ such that $\bar{q}(\theta)$ divides $\bar{g}_k(\theta)$. Then for $i=k,k+1,\ldots,n$ there exist distinct right ideals I_i and right T-module monomorphisms ψ_i: $T/Q\to T/I_i$ such that:*

(i) $M^2T \subseteq I_i \subset P$,

(ii) I_i *does not contain* MT,

(iii) $\operatorname{Im} \psi_i = P/I_i$.

Proof. Let Δ_p be as in the previous lemma. Since $\bar{R}[\theta]$ is a principal ideal domain, there are bases $\{\hat{u}_1(\theta),..., \hat{u}_n(\theta)\}$ and $\{\hat{v}_1(\theta),..., \hat{v}_n(\theta)\}$ of $\widehat{MT}$, such that the matrix of Δ_p relative to these bases is a diagonal matrix, D. Moreover, if the diagonal entries are $\bar{g}_1(\theta),..., \bar{g}_n(\theta)$, then we may assume that $\bar{g}_i(\theta)$ divides $\bar{g}_{i+1}(\theta)$ for $i = 1, 2,..., n-1$. In other words, $\bar{p}(\theta I_n + U) = ADB$ for some invertible matrices A and B in $\operatorname{Mat}_n \bar{R}[\theta]$. Since A and B are invertible, $\det A$ and $\det B$ are units in $\bar{R}[\theta]$, so $\det A$, $\det B \in \bar{R}$. We may thus assume that $\det A = \det B = 1$, and hence $\det \bar{p}(\theta I_n + U) = \bar{g}_1(\theta) \cdots \bar{g}_n(\theta)$.

From the above we have that $\Delta_p(\hat{u}_j(\theta)) = \hat{v}_j(\theta)\, \bar{g}_j(\theta)$. Thus

$$\operatorname{Im} \Delta_p = \hat{v}_1(\theta)\, \bar{g}_1(\theta)\, \bar{R}[\theta] \oplus \cdots \oplus \hat{v}_n(\theta)\, \bar{g}_n(\theta)\, \bar{R}[\theta].$$

By Lemma 3.5 $\operatorname{Im} \Delta_p = \bar{p}(\theta)\, \widehat{MT}$. For every j,

$$\hat{v}_j(\theta)\, \bar{g}_n(\theta) \in \hat{v}_j(\theta)\, \bar{g}_j(\theta)\, \bar{R}[\theta] \subseteq \bar{p}(\theta)\, \widehat{MT}.$$

Since $\{\hat{v}_1(\theta),..., \hat{v}_n(\theta)\}$ is a basis for $\widehat{MT}$, it follows that $\widehat{MT}\bar{g}_n(\theta) \subseteq \bar{p}(\theta)\, \widehat{MT}$. Let $\bar{h}(\theta)$ be such that $\widehat{MT}\bar{h}(\theta) \subseteq \bar{p}(\theta)\, \widehat{MT}$. In particular, $\hat{v}_n(\theta)\, \bar{h}(\theta) \in \operatorname{Im} \Delta_p$. It follows that $\bar{h}(\theta) = \bar{g}_n(\theta)\, \bar{f}(\theta)$ for some $\bar{f}(\theta) \in \bar{R}[\theta]$.

For $i = k, k+1,..., n$ let $\bar{g}_i = \bar{q}^l \bar{f}_i$, where $\bar{f}_i$ is relatively prime to $\bar{q}$. Let

$$I_i = M^2T + v_1 T + \cdots + v_{i-1} T + v_i q T + v_{i+1} T + \cdots + v_n T + pT.$$

Define $\psi_i'\colon T \to T/I_i$ by $\psi_i'(t) = v_i f_i t + I_i$. It is clear that $MT \subseteq \operatorname{Ker} \psi_i'$, and

$$\psi_i'(q) = v_i f_i q + I_i = v_i(f_i q - q f_i) + v_i q f_i + I_i = 0 + I_i$$

Hence $Q \subseteq \operatorname{Ker} \psi_i'$. We will show that $\operatorname{Ker} \psi_i' = Q$. Since Q is a maximal right ideal, it suffices to show that $\operatorname{Ker} \psi_i' \neq T$. If $v_i f_i \in I_i$, then

$$v_1 h_1 + \cdots + v_{i-1} h_{i-1} + v_i(f_i + q h_i) + v_{i+1} h_{i+1} + \cdots + v_n h_n = ph + m$$

for some $h_1,..., h_n, h \in T$ and $m \in M^2T$. From the last equation we have that $ph \in MT$. Since MT is a completely prime ideal and $p \notin MT$, we must have $h \in MT$. Therefore,

$$\hat{v}_1 \bar{h}_1 + \cdots + \hat{v}_{i-1} \bar{h}_{i-1} + \hat{v}_i(\bar{f}_i + \bar{q}\bar{h}_i) + \hat{v}_{i+1} \bar{h}_{i+1} + \cdots + \hat{v}_n \bar{h}_n \in \bar{p}\widehat{MT}.$$

Recall that $\hat{v}_1\, \bar{g}_1,..., \hat{v}_n\, \bar{g}_n$ is a basis for $\bar{p}\widehat{MT}$, so $\bar{f}_i + \bar{q}\bar{h}_i = \bar{g}_i \bar{t}$ for some $t \in T$. This is impossible since $\bar{f}_i$ and $\bar{q}$ are relatively prime. Therefore $\operatorname{Ker} \psi_i' = Q$.

Thus ψ_i' induces a monomorphism ψ_i of right T-modules, ψ_i: $T/Q \to T/I_i$, where $\psi_i(t+Q) = v_i f_i + I_i$.

I_i does not contain MT, because I_i does not contain $v_i f_i$. It is clear that $\operatorname{Im} \psi_i \subseteq P/I_i$. Let $\bar{r}_1, \bar{r}_2$ be such that $\bar{f}_i \bar{r}_1 + \bar{q} r_2 = 1$. Then $\psi_i(r_1 + Q) = v_i f_i r_1 + I_i = v_i + I_i$. Thus $v_j + I_i \in \operatorname{Im} \psi_i$ for $j = 1, 2,..., n$. This shows that $\operatorname{Im} \psi_i = P/I_i$. ∎

THEOREM 3.7. *Let R be a communtative noetherian ring and let $T = R[\theta; \delta]$. Let M be a maximal ideal of R such that $\delta(R) \subseteq M$, and let U be a matrix for the linear transformation $\hat{\delta}$. Let $P = MT + p(\theta)\, T$ be maximal ideal of T, and let Q be a prime ideal of T different from P. Then P is linked to itself, and P is linked to Q if and only if $Q = MT + q(\theta)\, T$ and $\bar{q}(\theta)$ is an irreducible factor of* $\det \bar{p}(\theta I_n + U)$.

Proof. The remarks following Lemma 3.4 show that P is linked to itself. It was shown in Proposition 3.3 that the conditions of this theorem are necessary for a link to exist. Let $\bar{q}(\theta)$ be an irreducible factor of $\det \bar{p}(\theta I_n + U)$, and let $Q = MT + q(\theta)\, T$. According to part 3 of the previous proposition, there is a right ideal I (let $I = I_n$) and a right T-module monomorphism ψ: $T/Q \to T/I$ such that the image of ψ is P/I. Thus we get an exact sequence of right T-modules;

$$0 \longrightarrow T/Q \xrightarrow{\psi} T/I \xrightarrow{\pi} T/P \longrightarrow 0$$

where π is the canonical map.

Since M annihilates both T/P and T/Q, but does not annihilate T/I this sequence cannot split. We can thus use Lemma 3.4, and conclude that P is linked to Q. ∎

LEMMA 3.8. *Let F be a field and U an $n \times n$ matrix over F, with eigenvalues $\lambda_1,..., \lambda_n$ in the algebraic closure of F. Let $f(x)$ be a polynomial with coefficients in F. Then* $\det f(xI_n + U) = f(x + \lambda_1) \cdots f(x + \lambda_n)$.

Proof. Because $(\lambda I_n - U) = ((x + \lambda)\, I_n - (xI_n + U))$, λ is an eigenvalue of U if and only if $x + \lambda$ is an eigenvalue of $xI_n + U$. Therefore $f(xI_n + U)$ has eigenvalues $f(x + \lambda_1),..., f(x + \lambda_n)$, thus proving the lemma. ∎

Lemma 3.8 together with Theorem 3.7 show that if λ is an eigenvalue of δ, then $MT + p(\theta)\, T \rightsquigarrow MT + p(\theta + \lambda)\, T$. Also note that if zero is an eigenvalue of $\hat{\delta}$, then $\bar{p}(\theta)$ is a divisor of $\det \bar{p}(\theta I_n + U)$. However, $P = MT + p(\theta)\, T$ is linked to itself regardless of whether zero is an eigenvalue or not. We will see the significance of having zero as an eigenvalue in Theorem 5.6.

COROLLARY 3.9. *Let R be a commutative noetherian ring containing the rational numbers, and let $T = R[\theta; \delta]$. Let $P = MT + p(\theta)\, T$ be a prime ideal* *of T. Then P is linked only to itself, if and only if $\hat{\delta}$ is a nilpotent linear transformation.*

Proof. Because of Lemma 2.11, we may assume that P and M are maximal ideals. If $\hat{\delta}$ is nilpotent, all its eigenvalues are 0. Thus $\det \bar{p}(\theta I + U) = p(\theta)^n$. Conversely, if P is linked only to itself, then $\bar{p}(\theta+\lambda_1)\cdots\bar{p}(\theta+\lambda_n) = \bar{p}(\theta)^n$ where $\lambda_1,\dots,\lambda_n$ are the eigenvalues of $\hat{\delta}$ in the algebraic closure of $\bar{R}$. Let $\alpha_1,\dots,\alpha_m$ be the roots of $\bar{p}(\theta)$ in the algebraic closure of $\bar{R}$. Let λ denote any one of the eigenvalues of $\hat{\delta}$. Then $\alpha_1-\lambda,\dots,\alpha_m-\lambda$ are also roots of $\bar{p}(\theta)$. Since $\bar{p}(\theta)$ is irreducible (and the characteristic of $\bar{R}$ is zero), the roots of $\bar{p}(\theta)$ must be distinct. Thus there is a permutation σ on $\{1, 2,\dots, m\}$ such that $\alpha_i = \alpha_{\sigma(i)} - \lambda$. It is clear that for any $1 \leqslant k \leqslant m$, $\alpha_1 = \alpha_{\sigma^k(1)} - k\lambda$. Since $\sigma^m(1) = 1$ we have that $m\lambda = 0$. Thus $\lambda = 0$. ▮

COROLLARY 3.10. *Let R be a commutative noetherian ring containing the rational numbers, and let $T = R[\theta; \delta]$. The connected components of the link graph of T are either singletons or they are infinite.*

Proof. Let P be a prime ideal of T. In view of Lemma 2.11 and the remark following it, we may assume that $P = MT + p(\theta)T$ and M is a maximal ideal of R such that $\delta(R) \subseteq M$. Let $\lambda_1,\dots,\lambda_n$ be the eigenvalues of $\hat{\delta}$ in the algebraic closure of $\bar{R}$. If $\lambda_1 = \cdots = \lambda_n = 0$, then the connected component containing P is just $\{P\}$. Thus we assume that one of the eigenvalues is nonzero; say, $\lambda_1 \neq 0$. Define a sequence of polynomials; $\bar{F}_k(\theta) = \bar{p}(\theta + k\lambda_1)\cdots\bar{p}(\theta + k\lambda_n)$. Since $\bar{F}_k(\theta) = \det \bar{p}(\theta I + kU)$ where U is the matrix of $\hat{\delta}$, it is clear that $\bar{F}_k(\theta)$ is a polynomial over $\bar{R}$.

Let α be a root of $\bar{p}(\theta)$ in the algebraic closure of $\bar{R}$. Then $\alpha - k\lambda_1$ is a root of $\bar{F}_k(\theta)$. Since $\lambda_1 \neq 0$, and since the characteristic of $\bar{R}$ is zero, $\alpha - k\lambda_1 \neq \alpha - m\lambda_1$ if $m \neq k$. Therefore, the number of irreducible factors appearing in the polynomials $\bar{F}_1(\theta),\dots, \bar{F}_k(\theta),\dots$ must be infinite.

We will now show the following. Let m be any positive integer, and let $\bar{q}_m(\theta)$ be any irreducible factor of $\bar{F}_m(\theta)$. Then there are irreducible polynomials $\bar{q}_i(\theta)$, $i = 1, 2,\dots, m-1$, such that $\bar{q}_i(\theta)$ divides $\bar{F}_i(\theta)$, and if $Q_i = MT + q_i(\theta)T$, then $P \rightsquigarrow Q_1$ and $Q_i \rightsquigarrow Q_{i+1}$ for $i = 1,\dots, m-1$. Since the number of irreducible polynomials occurring as factors of the polynomials $\bar{F}_k(\theta)$ is infinite, and since both m and $q_m(\theta)$ are arbitrarily chosen, the above statement completes the proof.

$\bar{F}_k(\theta+\lambda_1)\cdots\bar{F}_k(\theta+\lambda_n) = \det \bar{F}_k(\theta I_n + U)$, and thus is a polynomial with coefficients in $\bar{R}$. Furthermore, since $\bar{F}_k(\theta) = \bar{p}(\theta + k\lambda_1)\cdots\bar{p}(\theta + k\lambda_n)$ it is easy to see that $\bar{F}_{k+1}(\theta)$ divides $\bar{F}_k(\theta+\lambda_1)\cdots\bar{F}_k(\theta+\lambda_n)$. We know that $\bar{q}_m(\theta)$ divides $\bar{F}_m(\theta)$, and $\bar{F}_m(\theta)$ divides $\bar{F}_{m-1}(\theta+\lambda_1)\cdots\bar{F}_{m-1}(\theta+\lambda_n)$. Thus there is an irreducible factor $\bar{q}_{m-1}(\theta)$ of $\bar{F}_{m-1}(\theta)$ such that $\bar{q}_m(\theta)$ divides $\bar{q}_{m-1}(\theta+\lambda_1)\cdots\bar{q}_{m-1}(\theta+\lambda_n)$. It follows that $Q_{m-1} \rightsquigarrow Q_m$. By repeating this process, we are able to construct $\bar{q}_1(\theta),\dots, \bar{q}_{m-2}(\theta)$. ▮

4. The Ore Condition

In this section we will show that if Ω is a clique in a differential operator ring, $R[\theta; \delta]$, where R is a commutative noetherian ring containing the rational numbers, then $\mathscr{C}(\Omega)$ is an Ore set. First we state two general results. The proof of the first proposition is identical to the proof of [19, Proposition 2.1(iv) $\Rightarrow$ (i)], so we will not include the proof here. We will use the following notation. Let S be a subset of a ring R, and let J be an ideal of R. We let S/J denote the image of S in R/J under the canonical map.

Proposition 4.1. *Let S be a multiplicatively closed subset of a right noetherian ring R. Let I be a right AR-ideal of R with the following property; if $s - r \in I$ and $s \in S$, then $r \in S$. If S/I^n is a right Ore set in R/I^n for $n = 1, 2, \ldots$, then S is a right Ore set in R.*

The following proposition will allow us to simply verify that S/I is a right Ore set of R/I, instead of having to check this for all powers of I.

Proposition 4.2. *Let S be a multiplicatively closed subset of a ring R. Let I be an ideal of R such that for all s in S there is s' in S such that $Is' \subseteq sI + I^2$. If S/I is a right Ore set in R/I, then S/I^n is a right Ore set of R/I^n for $n = 1, 2, \ldots$.*

Proof. We first show by induction on n that for all s in S there is s' in S such that $I^n s' \subseteq sI^n + I^{n+1}$. By assumption we have s_1 in S such that $Is_1 \subseteq sI + I^2$, and by the induction hypothesis there is s_2 in S such that $I^n s_2 \subseteq s_1 I^n + I^{n+1}$. Let $s' = s_2$. Then $I^{n+1} s'_{n+1} \subseteq Is_1 I^n + I^{n+2} \subseteq (sI + I^2) I^n + I^{n+2} \subseteq sI^{n+1} + I^{n+2}$.

Let $s \in S$ and $r \in R$. We must show that there is $s_n \in S$, $r_n \in R$ such that $sr_n - rs_n \in I^n$. By assumption this is true for $n = 1$. From what we proved above, there is $s' \in S$ such that $I^n s' \subseteq sI^n + I^{n+1}$. Thus if we assume $sr_n - rs_n \in I^n$, then $(sr_n - rs_n) s' \in sI^n + I^{n+1}$. Hence $s(r_n s' - v) - r(s_n s') \in I^{n+1}$. ∎

Let Ω be a clique in a differential operator ring, $R[\theta; \delta]$. The following lemma shows that when proving that $\mathscr{C}(\Omega)$ is an Ore set we can assume that $P \cap R$ is a maximal ideal for P in Ω.

Lemma 4.3. *Let $S \subseteq S_1$ be multiplicatively closed sets of a ring R, and assume S is a right Ore set of R. If $S_1 S^{-1}$ is a right Ore set of RS^{-1}, then S_1 is a right Ore set of R.*

Proof. Take $r \in R$ and $s \in S_1$. Then $rS_1 S^{-1} \cap RS^{-1} \neq \varnothing$. Thus $rs_1 t_1^{-1} =$

$sr_1t_2^{-1}$ for some $t_1, t_2 \in S$, $r_1 \in R$ and $s_1 \in S_1$. There is $r_3 \in R$ and $t_3 \in S$ such that $t_2r_3 = t_1t_3$. Hence $rs_1t_3 = sr_1r_3$. ∎

LEMMA 4.4. *Let R be a commutative ring with a derivation δ, and let M be a maximal ideal of R such that $\delta(R) \subseteq M$. Let X be a set of maximal ideals of $R[\theta; \delta]$ such that $P \cap R = M$ for all P in X. Then $s(\theta) \in \mathscr{C}(X)$ if and only if for all $p(\theta) \in R[\theta; \delta]$ such that $MT + p(\theta)\,T \in X$, we have that $\bar{p}(\theta)$ does not divide $\bar{s}(\theta)$.*

Proof. The above condition is clearly necessary for $s(\theta)$ to be a member of $\mathscr{C}(X)$. The condition is also sufficient because every prime ideal of $R[\theta; \delta]$ is completely prime [18, Corollary 2.6], so that $\mathscr{C}(P) = T - P$ for all prime ideals of $T = R[\theta; \delta]$. ∎

PROPOSITION 4.5. *Let R be a commutative noetherian ring with a derivation δ and let M be a maximal ideal of R such that $\delta(R) \subseteq M$. Let X be a set of maximal ideals of $R[\theta; \delta]$ such that $P \cap R = M$ for all P in X. Assume further that if P is linked to Q and if Q is in X, then P is in X. Then $\mathscr{C}(X)$ is a right Ore set in $R[\theta; \delta]$.*

Proof. We will let the ideal MT play the role of the ideal I in the first two propositions of this section. We know that MT is an AR-ideal. Let $s - r \in MT$ and $s \in \mathscr{C}(X)$. If there is $P \in X$ and $t \in T$ such that $rt \in P$, then $st = (s - r)\,t + rt \in P$. Hence $r \in \mathscr{C}(X)$.

To show that $\mathscr{C}(X)$ is a right Ore set it is now sufficient to show that for all $s(\theta) \in \mathscr{C}(X)$ there is $s'(\theta) \in \mathscr{C}(X)$ such that $MTs'(\theta) \subseteq s(\theta)\,MT + M^2T$. Restating this with the notation of Section 3, we must show that $\widehat{MT}\bar{s}'(\theta) \subseteq \bar{s}(\theta)\,\widehat{MT}$. According to Proposition 3.6 we can find $s'(\theta) \in T$ such that $\bar{s}'(\theta)$ divides $\det \bar{s}(\theta I_n + U)$ and $\widehat{MT}\bar{s}'(\theta) \subseteq \bar{s}(\theta)\,\widehat{MT}$. We must show that $s'(\theta) \in \mathscr{C}(X)$. Let $q(\theta) \in T$ be such that $MT + q(\theta)\,T \in X$, and assume $\bar{q}(\theta)$ divides $\bar{s}'(\theta)$. Then $\bar{q}(\theta)$ divides $\det \bar{s}(\theta I_n + U)$, and hence for some irreducible factor $\bar{p}(\theta)$ of $\bar{s}(\theta)$, the polynomial $\bar{q}(\theta)$ divides $\det \bar{p}(\theta I_n + U)$. But then by Theorem 3.7 the ideal $MT + p(\theta)\,T$ is linked to $MT + q(\theta)\,T$, and thus $MT + p(\theta)\,T \in X$. This contradicts the assumption that $s(\theta) \in \mathscr{C}(X)$. Therefore by Lemma 4.4 $s'(\theta) \in \mathscr{C}(X)$. ∎

COROLLARY 4.6. *Let R be a commutative noetherian ring containing the rational numbers, and Q a prime ideal of $R[\theta; \delta]$. Let $\vec{\Omega}_Q$ be the right clique of Q. Then $\mathscr{C}(\vec{\Omega}_Q)$ is a right Ore set of $R[\theta; \delta]$, and is the maximal right Ore set contained in $\mathscr{C}(Q)$.*

Proof. This follows immediately from Proposition 4.5 and Lemma 4.3. ∎

Since R is commutative, there is an anti-isomorphism ϕ of $R[\theta; \delta]$ such that $\phi(\theta) = -\theta$ and $\phi(r) = r$ for all r in R. It is clear that P is linked to Q if and only if $\phi(Q)$ is linked to $\phi(P)$. Thus $\phi(\underset{\rightarrow}{\Omega}_Q) = \{\phi(P) \mid P \in \underset{\rightarrow}{\Omega}_Q\} = {}_{\phi(Q)}\underset{\leftarrow}{\Omega}$, and hence $\mathscr{C}({}_Q\underset{\leftarrow}{\Omega})$ is a left Ore set for all prime ideals Q of $R[\theta; \delta]$. Moreover, we get a left sided version of Proposition 4.5 which in turn gives the following result.

COROLLARY 4.7. *Let R be a commutative noetherian ring containing the rational numbers, and let Ω be a clique in $R[\theta; \delta]$. Then $C(\Omega)$ is an Ore set in $R[\theta; \delta]$.*

5. MULTIPLICITIES

In Section 3 we showed that a maximal ideal P of a differential operator ring $T = R[\theta; \delta]$ is linked to a maximal ideal Q by producing an exact sequence

$$0 \longrightarrow T/Q \xrightarrow{\psi} T/I \xrightarrow{\pi} T/P \longrightarrow 0.$$

The image of the map ψ is P/I, so P/I is annihilated by Q. We can thus think of the module T/I as having a bottom part annihilated by Q, and a top part annihilated by P (after you factor out the bottom part P/I, you get T/P). It is also easy to check that P/I is an essential submodule of T/I. Thus $E_T(T/Q) \cong E_T(T/I)$, and it is now clear that there is a connection between the structure of the injective module $E_T(T/Q)$ and the primes P that are linked to Q.

We will follow [13], and describe how the multiplicity of a link between two prime ideals is defined. Let R be a noetherian ring, and let P and Q be prime ideals of R. Let U be a uniform right ideal of R/Q, and set $E_Q = E_R(U)$. Up to isomorphism, the injective module E_Q does not depend on the choice of the right ideal U, because any two right ideals in a prime noetherian ring are subisomorphic. Since U is uniform, E_Q is an indecomposable module.

In general, if E is an injective module over R, then there is a family $\{E_i \mid i \in I\}$ of mutually nonisomorphic indecomposable injective modules, and a family $\{\alpha_i \mid i \in I\}$ of nonzero cardinals such that $E = \bigoplus_{i \in I} E_i^{(\alpha_i)}$. This decomposition is unique up to a permutation of the set I [16, Proposition 2.7]. We now apply this decomposition, and write $E_R(E_Q/\operatorname{ann}_{E_Q} Q) = \bigoplus_{i \in I} E_i^{(\alpha_i)}$, where the E_i's and the α_i's are as above. We define the *multiplicity* of P in Q, or the *multiplicity of the link* between P and Q (denoted $\operatorname{mult}(P, Q)$), to be the cardinal α_i if $E_P \cong E_i$ for some i in I. If E_P is not isomorphic to any of the E_i's, then we set $\operatorname{mult}(P, Q) = 0$.

The point of the discussion at the beginning of the section was to show that if P and Q are maximal ideals in a differential operator ring $T = R[\theta; \delta]$, and P is linked to Q, then $\operatorname{mult}(P, Q) > 0$.

We will use the following lemma proved in [13, Lemma 6.1] to show that if P is linked to Q, then $\operatorname{mult}(P, Q) > 0$.

LEMMA 5.1. *Let Q be a prime ideal in a noetherian ring R, and let J and P be ideals of R such that $PJ = 0$ and $JQ = 0$. Further assume that J is a torsion-free right R/Q-module. Let $F = \operatorname{ann}_{E_Q} J$. Then $(E_Q/F)\, P = 0$ and E_Q/F is isomorphic to* $\operatorname{Hom}_R(J, E_Q)$. *Moreover, an element d of R is a nonzero divisor on the right R-module E_Q/F if and only if d is a nonzero divisor on the left R-module J.*

LEMMA 5.2. *Let P and Q be distinct prime ideals of a noetherian ring R. Then* $\operatorname{ann}_{E_Q} Q = \operatorname{ann}_{E_Q} QP$.

Proof. $E_Q = E_R(U)$ where $U = U'/Q$ is a uniform right ideal of R/Q. Take $x \in \operatorname{ann}_{E_Q} QP$, so $xQP = 0$. Since $P \neq Q$ we have that $(P + Q) \cap \mathscr{C}(Q) \neq \varnothing$ and thus there is $c \in \mathscr{C}(Q) \cap P$. Hence $xQc = 0$. If $xQ \neq 0$, then $xQ \cap U \neq 0$ since E_Q is a uniform R-module. Thus there is $u \in U'$, $u \notin Q$ such that $uc \in Q$ contradicting the choice of c. Therefore $x \in \operatorname{ann}_{E_Q} Q$. ∎

Note that since $QP \subseteq P \cap Q \subseteq Q$, Lemma 5.2 shows that $\operatorname{ann}_{E_Q} Q = \operatorname{ann}_{E_Q} P \cap Q$.

LEMMA 5.3. *Let P and Q be prime ideals in a noetherian ring R, and assume P is linked to Q. Then* $\operatorname{mult}(P, Q) > 0$.

Proof. Since $P \rightsquigarrow Q$ there is an ideal I such that $PQ \subseteq I \subset P \cap Q$, and $P \cap Q/I$ is torsion-free as a right R/Q-module and as a left R/P-module. By first factoring out I, we may assume that $I = 0$. We now apply Lemma 5.1 with $J = P \cap Q$. By Lemma 5.2, $F = \operatorname{ann}_{E_Q} J = \operatorname{ann}_{E_Q} Q$. It is easy to check that $\operatorname{Hom}_R(P \cap Q, E_Q) \neq 0$; therefore $E_Q/F \neq 0$. Furthermore, since $J = P \cap Q$ is torsion-free as a left R/P-module, E_Q/F is torsion-free as a right R/P-module. Let M be a uniform submodule of E_Q/F. Then $MP = 0$ and M is torsion-free as an R/P-module. It follows that $E_P \cong E_R(M)$. Thus $\operatorname{mult}(P, Q) > 0$. ∎

For an AR-separated noetherian ring R, the converse of this Lemma holds [13, Lemma 6.3]. The differential operator rings discussed in this paper are AR-separated. Thus if we know the multiplicities of the links, we will have precise information about the structure of E_Q for any prime ideal Q. Intuitively, the module E_Q can be thought of as being made up of layers. The first layer is $\operatorname{ann}_{E_Q} Q$. After we factor out $\operatorname{ann}_{E_Q} Q$ the injective hull of the resulting module can be decomposed into a direct sum of indecomposable injectives. Each injective in this decomposition is isomorphic to E_P for some prime P linked to Q, and if P is linked to Q, then E_P appears in the decomposition. Moreover, the multiplicity of the link between P and Q tells us how many copies of the module E_P there are in the decomposition.

The following lemma tells us that when computing the multiplicities of the links in a differential operator ring, we can reduce to the case of maximal ideals.

LEMMA 5.4. *Let R be a noetherian ring, Q a prime ideal and S a right Ore set of R disjoint from Q. Then $E_Q = E_{QS^{-1}}$.*

Proof. Let U be a uniform right ideal of R/Q, so that $E_Q = E_R(U)$. Let $\bar{S} = S/Q$, and note that $\bar{S}$ is a right Ore set of R/Q consisting of regular elements (Lemma 2.9). Hence $U\bar{S}^{-1}$ is a uniform right ideal of $(R/Q)\,\bar{S}^{-1}$, and thus $E_{QS^{-1}} = E_{RS^{-1}}(U\bar{S}^{-1})$. It is now easy to see that U is an essential R-submodule of $E_{QS^{-1}}$. On the other hand, RS^{-1} is a flat R-module, so $E_{QS^{-1}}$ is an injective R-module. Hence $E_{QS^{-1}} = E_Q$. ∎

In [13, Theorem 6.12] a formula for the multiplicities is derived. It is fairly easy to prove this formula for our particular situation.

PROPOSITION 5.5. *Let R be a noetherian ring and let P and Q be maximal ideals of R, such that R/P and R/Q are division rings. Let $V = P \cap Q/PQ$ and assume V is nonzero, and finite dimensional both as a left R/P vector space and as a right R/Q vector space. Let $S = \operatorname{End} V_{R/Q}$, and let I be a minimal right ideal of S. Then* $\operatorname{mult}(P, Q) = \dim I_{R/P}$

Proof. Let $F = \operatorname{ann}_{E_Q} Q$, and let $A = \{x \in E_Q \mid xP \subseteq F\}$. Then $\operatorname{mult}(P, Q)$ is the dimension of A/F as a vector space over R/P. By applying $\operatorname{Hom}_R(\ , E_Q)$ to the exact sequence

$$0 \longrightarrow V \longrightarrow R/PQ \longrightarrow R/P \cap Q \longrightarrow 0$$

we get the exact sequence

$$0 \rightarrow \operatorname{Hom}_R(R/P \cap Q, E_Q) \rightarrow \operatorname{Hom}_R(R/PQ, E_Q) \rightarrow \operatorname{Hom}_R(V, E_Q) \rightarrow 0.$$

The map $f \mapsto f(1 + PQ)$ identifies $\operatorname{Hom}_R(R/PQ, E_Q)$ with A, and the map $f \mapsto f(1 + P \cap Q)$ identifies $\operatorname{Hom}_R(R/P \cap Q, E_Q)$ with F (the reason $f(1 + P \cap Q)$ is a member of F is that $F = \operatorname{ann}_{E_Q} P \cap Q$). Thus $A/F \cong \operatorname{Hom}_R(V, E_Q)$. If $f \in \operatorname{Hom}_R(V, E_Q)$, then $\operatorname{Im} f \subseteq F$. Thus $\operatorname{Hom}_R(V, E_Q) = \operatorname{Hom}_R(V, F) = \operatorname{Hom}_{R/Q}(V, F)$. Since $\operatorname{Hom}_R(V, F)$ is a simple right S-module, and S is a simple artinian ring, $\operatorname{Hom}_R(V, F) \cong I$, thus proving the proposition. ∎

As before let $T=R[\theta;\delta]$ where R is a commutative noetherian ring. Let M be a maximal ideal of R such that $\delta(R)\subseteq M$, and let P and Q be maximal ideals of T such that P is linked to Q and $P\cap R=Q\cap R=M$. Let $V=P\cap Q/PQ$. It is clear that T/P and T/Q are fields, and V is a left vector space over T/P and a right vector space over T/Q. It is also easy to see that T/P embeds into End $V_{T/Q}$ as left multiplications. Thus if I is a minimal right ideal of End $V_{T/Q}$, then I can be considered as a right vector space over T/P. We will now apply Proposition 5.5, and show how mult(P, Q) can be computed from the information obtained in Section 3. Note that T/P and T/Q are both finite field extensions of the field R/M. Moreover, $cv=vc$ for all c in R/M and v in V. Hence

$$\dim_{R/M} V=\dim V_{R/M}=\dim V_{T/Q}[T/Q:R/M]$$

and

$$\dim_{R/M} V= \dim I_{R/M}=\dim I_{T/P}[T/P:R/M].$$

We now get the following result.

THEOREM 5.6. *Let $T=R[\theta;\delta]$, where R is a commutative noetherian ring. Let M be a maximal ideal of R such that $\delta(R)\subseteq M$. Let $P=MT+p(\theta)T$ and $Q=MT+q(\theta)T$ be maximal ideals of T, and let U be a matrix for $\hat{\delta}$. Write* $\det \bar{p}(\theta I_n+U)=\bar{g}_1\cdots\bar{g}_n$, *where $\bar{g}_1,\ldots,\bar{g}_n$ are the polynomials described in Proposition* 3.6. *Let k be the least integer i such that $\bar{q}$ divides $\bar{g}_i$ (if $\bar{q}$ divides none of the $\bar{g}_i$'s we let $k=n+1$). Then*

$$\operatorname{mult}(P,Q)=\begin{cases}\dfrac{\deg\bar{q}}{\deg\bar{p}}(n-k+1) & \text{if } P\neq Q\\[2ex] n-k+2 & \text{if } P=Q.\end{cases}$$

Remark. Note that $n-k+1$ is the number of different $\bar{g}_i$'s the polynomial $\bar{q}$ divides.

Proof. Let $W=(MT+pqT)/(M^2T+MqT+pMT+pqT)$, and note that when $P\neq Q$, then $V=W$ (Lemma 3.1). If $P=Q$, then

$$V=P/P^2=(MT+pT)/(M^2T+MpT+pMT+p^2T)$$

and

$$W=(MT+p^2T)/(M^2T+MpT+pMT+p^2T).$$

It is clear in this case that W is a subspace of V of codimension 1. Since $[T/Q:R/M]=\deg\bar{q}$ and $[T/P:R/M]=\deg\bar{p}$, the above discussion together with the remarks preceding this theorem show that it suffices to prove that $\dim W_{T/Q} = n - k + 1$.

It is easy to see that $W \cong \widehat{MT}/\bar{p}\widehat{MT} + \widehat{MT}\bar{q}$. Recall that, with the notation of Proposition 3.6,

$$\widehat{MT} = \bigoplus_{i=1}^{n} \hat{v}_i \bar{R}[\theta] \qquad \text{and} \qquad \bar{p}\widehat{MT} = \bigoplus_{i=1}^{n} \hat{v}_i \bar{g}_i \bar{R}[\theta].$$

Hence $\widehat{MT}\bar{q} = \bigoplus_{i=1}^{n} \hat{v}_i \bar{q} \bar{R}[\theta]$ and

$$\hat{v}_i \bar{q} \bar{R}[\theta] \oplus \hat{v}_i \bar{g}_i \bar{R}[\theta] = \begin{cases} \hat{v}_i \bar{q} \bar{R}[\theta] & \text{if } \bar{q} \text{ divides } \bar{g}_i \\ \hat{v}_i \bar{R}[\theta] & \text{if } \bar{q} \text{ does not divide } \bar{g}_i. \end{cases}$$

Therefore $\bar{p}\widehat{MT} + \widehat{MT}\bar{q} = \bigoplus_{i=1}^{k-1} \hat{v}_i \bar{R}[\theta] \oplus \bigoplus_{i=k}^{n} \hat{v}_i \bar{q} \bar{R}[\theta]$. The result is now clear. ∎

We conclude this section by computing the links, and the multiplicities of the links for a few concrete examples.

EXAMPLES 5.7. Let F be a field of characteristic zero, and let $R = F[x]$. Let δ be a nonzero derivation on R such that $\delta(F) = 0$. Then $\delta = f(x)(d/dx)$ for some nonzero $f \in R$. Let $T = R[\theta; \delta]$, and let P be a nonzero prime ideal of T. Lemma 1.6 implies that $P \cap R \neq 0$.

Since $P \cap R$ is both a δ-ideal and a maximal ideal of R, we must have that $P \cap R = gR$ where g is an irreducible factor of f. Hence if $P \neq gT$, then $P = gT + p(\theta) T$ for some $p(\theta)$ in T. Since g divides f, we can write $f = hg^2 + rg$ for some $h, r \in R$ and $\deg r < \deg g$. With our previous notation $M = gR$ and (r) is a matrix for $\hat{\delta}$. We have now completely determined the link graph of $R[\theta; \delta]$.

(i) The zero ideal has no links.

(ii) For each irreducible factor g of f, the ideal gT is a prime ideal which is linked only to itself.

(iii) For each of the prime ideals gT, where g is an irreducible factor of f, there are infinitely many maximal ideals of the form $P = gT + p(\theta) T$ containing gT. Each of these ideals are linked to themselves, and if g^2 divides f, then these are the only links these ideals have. If g^2 does not divide f, then $r \neq 0$ and $gT + p(\theta) T \rightsquigarrow gT + q(\theta) T$ if and only if $q(\theta) = p(\theta + r)$.

The above is a complete list of all the prime ideals of $R[\theta; \delta]$, and describes completely the link graph of $R[\theta; \delta]$. It is clear that all of these links have multiplicity 1.

EXAMPLE 5.8. Let $R = \mathbb{C}[x_1, ..., x_n]$, and let δ be a derivation on R. Let M be a maximal ideal of R, such that $\delta(R) \subseteq M$, and let P be a maximal ideal of $T = R[\theta; \delta]$ such that $P \cap R = M$. Then $M = (x_1 - c_1) R + \cdots + (x_n - c_n) R$ for some $c_1, ..., c_n \in \mathbb{C}$. Since $R/M \cong \mathbb{C}$, we have that $P = MT + (\theta - a) T$ for some $a \in \mathbb{C}$. Clearly $x - c_1 + M^2, ..., x - c_n + M^2$ is a basis for $\hat{M}$, so dim $\hat{M} = n$. Let $\lambda_1, ..., \lambda_k$ be the distinct nonzero eigenvalues of $\hat{\delta}$, and let $Q_i = MT + (\theta - a + \lambda_i) T$. Then P is linked to Q if and only if $Q = P$ or $Q = Q_i$ for some i. Moreover, mult(P, Q_i) is the dimension of the eigenspace of λ_i, and mult(P, P) is 1 plus the dimension of the kernel of $\hat{\delta}$.

EXAMPLE 5.9. Let $R = \mathbb{C}[x, y]$, and let $M = xR + yR$. Let $\delta = x(\partial/\partial x) + iy(\partial/\partial y)$. Then the eigenvalues of $\hat{\delta}$ are 1 and i. The maximal ideal $MT + (\theta + a) T$, where a is a complex number, is linked to itself and the ideals $MT + (\theta + a + 1) T$ and $MT + (\theta + a + i) T$. Let $Q = MT + \theta T$. Then

$$\underset{\rightarrow}{\Omega}_Q = \{MT + (\theta + n + mi) T \mid n, m \leqslant 0 \text{ and } n, m \in \mathbb{Z}\},$$

$${}_Q\underset{\rightarrow}{\Omega} = \{MT + (\theta + n + mi) T \mid n, m \geqslant 0 \text{ and } n, m \in \mathbb{Z}\}.$$

This is an example where the clique of a prime is not the union of its right and left cliques, because the clique of Q is $\{MT + (\theta + n + mi) T \mid n, m \in \mathbb{Z}\}$.

EXAMPLE 5.10. Let $R = \mathbb{R}[x, y]$ and let M be a maximal ideal of R such that $R/M \cong \mathbb{R}$. Let δ be a derivation on R such that $\delta(R) \subseteq M$, and assume the characteristic polynomial of $\hat{\delta}$ is irreducible over $\mathbb{R}$. Let U be a matrix for $\hat{\delta}$ and let $f(x) = \det(xI + U)$. Then $f(x)$ is an irreducible polynomial over $\mathbb{R}$, whence $f(x) = (x + a)^2 + b^2$ where a and b are real numbers and b is positive. Let $T = R[\theta; \delta]$. The maximal ideals of T whose intersection with R equals M, are of two types:

$$P_\alpha = MT + (\theta + \alpha) T \qquad \text{and} \qquad Q_{\alpha,\beta} = MT + ((\theta + \alpha)^2 + \beta^2) T,$$

where α and β are real numbers and β is positive. If $p(\theta) = \theta + \alpha$, then

$$\det p(\theta I + U) = \det((\theta + \alpha) I + U) = f(\theta + \alpha) = (\theta + \alpha + a)^2 + b^2.$$

Thus $P_\alpha \rightsquigarrow Q_{\alpha + a, b}$. Since $\det p(\theta I + U)$ is, in this case, an irreducible polynomial, it follows from Theorem 5.6 that mult$(P_\alpha, Q_{\alpha + a, b}) = 2$. If $q(\theta) = (\theta + \alpha)^2 + \beta^2$, then

$$\begin{aligned} \det p(\theta I + U) &= \det((\theta + \alpha + i\beta) I + U) \det((\theta + \alpha - i\beta) I + U) \\ &= f(\theta + \alpha + i\beta) f(\theta + \alpha - i\beta) \\ &= ((\theta + \alpha + a)^2 + (\beta + b)^2)((\theta + \alpha + a)^2 + (\beta - b)^2). \end{aligned}$$

Thus $Q_{\alpha,\beta} \rightsquigarrow Q_{\alpha + a, \beta + b}$. If $\beta \neq b$, then $Q_{\alpha,\beta} \rightsquigarrow Q_{\alpha + a, \beta - b}$. If $\beta = b$, then $Q_{\alpha,b} \rightsquigarrow P_{\alpha + a}$. It is clear that all these links have multiplicity 1.

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